ON CERTAIN C-TEST WORDS FOR FREE GROUPS

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Abstract. Let F_m be a free group of a finite rank $m \geq 2$ and X_i, Y_j be elements in F_m. A non-empty word $w(x_1, \ldots, x_n)$ is called a C-test word in n letters for F_m if, whenever $w(X_1, \ldots, X_n) = w(Y_1, \ldots, Y_n) \neq 1$, the two n-tuples $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ are conjugate in F_m. In this paper we construct, for each $n \geq 2$, a C-test word $v_n(x_1, \ldots, x_n)$ with the additional property that $v_n(X_1, \ldots, X_n) = 1$ if and only if the subgroup of F_m generated by $X_1, \ldots, X_n$ is cyclic. Making use of such words $v_m(x_1, \ldots, x_m)$ and $v_{m+1}(x_1, \ldots, x_{m+1})$, we provide a positive solution to the following problem raised by Shpilrain: There exist two elements $u_1, u_2 \in F_m$ such that every endomorphism ψ of F_m with non-cyclic image is completely determined by $\psi(u_1)$, $\psi(u_2)$.

1. Introduction

Let $F_m = \langle x_1, \ldots, x_m \rangle$ be the free group of a finite rank $m \geq 2$ on the set $\{x_1, \ldots, x_m\}$. The purpose of this paper is to present a positive solution to the following problem raised by Shpilrain [1]:

Problem. Are there 2 elements u_1, u_2 in F_m such that any endomorphism ψ of F_m with non-cyclic image is uniquely determined by $\psi(u_1)$, $\psi(u_2)$? (In other words, are there 2 elements u_1, u_2 in F_m such that whenever $\phi(u_i) = \psi(u_i)$, $i = 1, 2$, for endomorphisms ϕ, ψ of F_m with non-cyclic images, it follows that $\phi = \psi$?)

In [2], Ivanov solved in the affirmative this problem in the case where ψ is a monomorphism of F_m by constructing a so-called C-test word $w_n(x_1, \ldots, x_n)$ for each $n \geq 2$. A C-test word is defined due to Ivanov [2] as follows:

Definition. A non-empty word $v(x_1, \ldots, x_n)$ is a *C-test word* in n letters for F_m if for any two n-tuples $(X_1, \ldots, X_n)$, $(Y_1, \ldots, Y_n)$ of elements of F_m the equality $v(X_1, \ldots, X_n) = v(Y_1, \ldots, Y_n) \neq 1$ implies the existence of an element $S \in F_m$ such that $Y_i = SX_iS^{-1}$ for all $i = 1, 2, \ldots, n$.

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According to the result of [2, Corollary 1], if v is a C-test word in m letters for F_m, ϕ is an endomorphism, ψ is a monomorphism of F_m, and $\phi(v) = \psi(v)$, then we have $\phi = \tau_S \circ \psi$, where $S \in F_m$ is such that $\langle S, \psi(v)\rangle$ is cyclic, and τ_S is the inner automorphism of F_m defined by means of S. Notice that this assertion is no longer true if ψ is extended to an endomorphism of F_m with non-cyclic image, since the fact $\psi(v) \neq 1$ is no longer guaranteed. The efforts to extend the result of [2, Corollary 1] to the case where ψ is an endomorphism of F_m with non-cyclic image have led us to proving the following:

Theorem. *For every $n \geq 2$ there exists a C-test word $v_n(x_1, \ldots, x_n)$ in n letters for F_m with the additional property that $v_n(X_1, \ldots, X_n) = 1$ if and only if the subgroup $\langle X_1, \ldots, X_n\rangle$ of F_m generated by $X_1, \ldots, X_n$ is cyclic.*

We construct such a C-test word $v_n(x_1, \ldots, x_n)$ by combining Ivanov's C-test word $w_2(x_1, x_2)$ and an auxiliary word $u(x_1, x_2)$ defined below. Here, let us recall Ivanov's C-test word $w_2(x_1, x_2)$:

$$\begin{aligned} w_2(x_1, x_2) = {} & [x_1^8, x_2^8]^{100} x_1 [x_1^8, x_2^8]^{200} x_1 [x_1^8, x_2^8]^{300} x_1^{-1} [x_1^8, x_2^8]^{400} x_1^{-1} \\ & [x_1^8, x_2^8]^{500} x_2 [x_1^8, x_2^8]^{600} x_2 [x_1^8, x_2^8]^{700} x_2^{-1} [x_1^8, x_2^8]^{800} x_2^{-1}. \end{aligned}$$

We define an auxiliary word $u(x_1, x_2)$ as follows:

$$\begin{aligned} u(x_1, x_2) = {} & [x_1^{48}, x_2^{40}]^{100} x_1^6 [x_1^{48}, x_2^{40}]^{200} x_1^6 [x_1^{48}, x_2^{40}]^{300} x_1^6 [x_1^{48}, x_2^{40}]^{400} x_1^6 \\ & [x_1^{48}, x_2^{40}]^{500} x_2^5 [x_1^{48}, x_2^{40}]^{600} x_2^5 [x_1^{48}, x_2^{40}]^{700} x_2^5 [x_1^{48}, x_2^{40}]^{800} x_2^5. \end{aligned} \tag{1.1}$$

We then construct $v_n(x_1, \ldots, x_n)$ as follows: If $n = 2$ then

$$v_2(x_1, x_2) = w_2(x_1, x_2). \tag{1.2}$$

If $n = 3$ then

$$v_3(x_1, x_2, x_3) = u\Big(u\big(v_2(x_1, x_2), v_2(x_2, x_3)\big),\ u\big(v_2(x_2, x_3), v_2(x_3, x_1)\big)\Big). \tag{1.3}$$

Inductively, for $n \geq 4$, define

$$v_n(x_1,\dots,x_n) = u\Big(u\big(v_{n-1}(x_1,x_2,x_3,\dots,x_{n-1}),\ v_{n-1}(x_{n-1},x_2,x_3,\dots,x_{n-2},x_n)\big),\\ u\big(v_{n-1}(x_{n-1},x_2,x_3,\dots,x_{n-2},x_n),\ v_{n-1}(x_n,x_2,x_3\dots,x_{n-2},x_1)\big)\Big). \tag{1.4}$$

For instance,

$$v_4(x_1,x_2,x_3,x_4) = u\Big(u\big(v_3(x_1,x_2,x_3),v_3(x_3,x_2,x_4)\big),\ u\big(v_3(x_3,x_2,x_4),v_3(x_4,x_2,x_1)\big)\Big). \tag{1.5}$$

In Section 2, we establish several technical lemmas concerning properties of Ivanov's word $v_2(x_1,x_2)$ and the auxiliary word $u(x_1,x_2)$ which will be used throughout this paper. In Sections 3–5, we prove that, for each $n \geq 3$, the word $v_n(x_1,\dots,x_n)$ constructed above is indeed a C-test word with the property in the statement of the Theorem (the case $n = 2$ is already proved in [2]). We first treat the case $n = 3$ in Section 3, and then proceed by simultaneous induction on n with base $n = 4$ together with several necessary lemmas in Sections 4–5.

Once the Theorem is proved, our Corollary 1 that is an extended version of [2, Corollary 1] to the case where ψ is an endomorphism of F_m with non-cyclic image follows immediately, as intended, by taking $u = v_m(x_1,\dots,x_m)$:

Corollary 1. *There exists an element $u \in F_m$ such that if ϕ is an endomorphism, ψ is an endomorphism of F_m with non-cyclic image, and $\phi(u) = \psi(u)$, then ϕ also has non-cyclic image, more precisely, $\phi = \tau_S \circ \psi$, where $S \in F_m$ is such that $\langle S, \psi(u)\rangle$ is cyclic, and τ_S is the inner automorphism of F_m defined by means of S.*

In Corollary 2, we provide a positive solution to the Shpilrain's problem mentioned above:

Corollary 2. *There exist two elements $u_1, u_2 \in F_m$ such that any endomorphism ψ of F_m with non-cyclic image is uniquely determined by $\psi(u_1)$, $\psi(u_2)$.*

The proof of Corollary 2 makes use of the words $v_m(x_1,\dots,x_m)$ and $v_{m+1}(x_1,\dots,x_{m+1})$. Its detailed proof is given in Section 6. The idea and the techniques used in [2] are developed further in the present paper.

2. Preliminary Lemmas

We begin this section by establishing some notation and terminology. We let X, Y (with or without subscript) be words in F_m throughout this paper. By $X = Y$ we denote the equality in F_m of words X and Y, and by $X \equiv Y$ the graphical (letter-by-letter) equality of words X and Y. The length of a word X is denoted by $|X|$ (note $|x_1 {x_1}^{-1}| = 2$). We say that a word X is a *proper power* if $X = Y^\ell$ for some Y with $\ell > 1$ and that a word A is *simple* if A is non-empty, cyclically reduced, and is not a proper power. If A is simple, then an *A-periodic* word is a subword of A^k with some $k > 0$.

Now let us introduce several lemmas concerning properties of the word $v_2(x_1, x_2)$ defined by (1.2). For proofs of Lemmas 1–2, see [2].

Lemma 1 [2, Lemma 3]. *If the subgroup $\langle X_1, X_2\rangle$ of F_m is non-cyclic, then $v_2(X_1, X_2)$ is neither equal to the empty word nor a proper power. If $\langle X_1, X_2\rangle$ is cyclic, then $v_2(X_1, X_2) = 1$.*

Lemma 2 [2, Lemma 4]. *If the subgroup $\langle X_1, X_2\rangle$ of F_m is non-cyclic and $v_2(X_1, X_2) = v_2(Y_1, Y_2)$, then there exists a word $Z \in F_m$ such that*

$$Y_1 = ZX_1Z^{-1} \quad \text{and} \quad Y_2 = ZX_2Z^{-1}.$$

Lemma 3. *If the subgroup $\langle X_1, X_2\rangle$ of F_m is non-cyclic, then $v_2(X_1, X_2)^{-1} \neq v_2(Y_1, Y_2)$ for any words Y_1, Y_2.*

Proof. By way of contradiction, suppose that $v_2(X_1, X_2)^{-1} = v_2(Y_1, Y_2)$ for some words Y_1, Y_2. If $\langle Y_1, Y_2\rangle$ is cyclic, then it follows from Lemma 1 that $v_2(Y_1, Y_2) = 1$, so that $v_2(X_1, X_2) = 1$, i.e., $\langle X_1, X_2\rangle$ is cyclic. This contradiction to the hypothesis of the lemma allows us to assume that $\langle Y_1, Y_2\rangle$ is non-cyclic. As in [2, Lemmas 1–4], let W be a cyclically reduced word that is conjugate to $v_2(X_1, X_2)$, and let B be a simple word such that $[X_1^8, X_2^8]$ is conjugate to B (recall from [3, 4] that a commutator $[A, B]$ of two words A, B is not a proper power). Then according to [2,

Lemma 2], W has the form

$$W \equiv R_1 T_1 R_2 T_2 \cdots R_8 T_8,$$

where R_i are B-periodic words with $(i \cdot 100 - 14)|B| < |R_i| \le i \cdot 100|B|$, $0 \le |T_i| < 6|B|$, and $3488|B| < |W| < 3648|B|$. The same holds for $v_2(Y_1, Y_2)$ and attach the prime sign $'$ to the notations for $v_2(Y_1, Y_2)$. Then the equality $v_2(X_1, X_2)^{-1} = v_2(Y_1, Y_2)$ yields that W^{-1} is a cyclic permutation of W', so that $|W^{-1}| = |W'|$. At this point, apply the arguments in [2, Lemma 4] to W^{-1}, W' to get $B^{-1} \equiv B'$ and that R_i' B'-overlaps only with R_i^{-1} for each $i = 1, 2, \ldots, 8$. But this is impossible, for if R_i' B'-overlapped only with R_i^{-1} then R_{i+1}' would have to B'-overlap only with R_{i-1}^{-1} (indices modulo 8) by the order of indices in $W^{-1} \equiv T_8^{-1} R_8^{-1} \cdots T_2^{-1} R_2^{-1} T_1^{-1} R_1^{-1}$ and $W' \equiv R_1' T_1' R_2' T_2' \cdots R_8' T_8'$. □

We also establish several lemmas concerning properties of the auxiliary word $u(x_1, x_2)$ defined by (1.1).

Lemma 4. *If the subgroup $\langle X_1, X_2 \rangle$ of F_m is non-cyclic, then $u(X_1, X_2)$ is neither equal to the empty word nor a proper power. If $\langle X_1, X_2 \rangle$ is cyclic, then either $u(X_1, X_2)$ is equal to the empty word provided $X_1^6 = X_2^{-5}$ or otherwise $u(X_1, X_2)$ is a proper power.*

Proof. The proof of the first part is similar to that of [2, Lemma 3], and the second part is immediate from definition (1.1) of $u(x_1, x_2)$. □

Lemma 5. *If the subgroup $\langle X_1, X_2 \rangle$ of F_m is non-cyclic and $u(X_1, X_2) = u(Y_1, Y_2)$, then there exists a word $Z \in F_m$ such that*

$$Y_1 = Z X_1 Z^{-1} \quad \textit{and} \quad Y_2 = Z X_2 Z^{-1}.$$

Proof. Applying the same arguments as in [2, Lemma 4] to $u(X_1, X_2)$ and $u(Y_1, Y_2)$, we deduce that $Y_1^6 = Z X_1^6 Z^{-1}$ and $Y_2^5 = Z X_2^5 Z^{-1}$ for some word $Z \in F_m$. Since extraction of roots is unique in a free group, it follows that $Y_1 = Z X_1 Z^{-1}$ and $Y_2 = Z X_2 Z^{-1}$, as required. □

Lemma 6. *If the subgroup $\langle X_1, X_2\rangle$ of F_m is non-cyclic, then $u(X_1, X_2)^{-1} \neq u(Y_1, Y_2)$ for any words Y_1, Y_2.*

Proof. Suppose to the contrary that $u(X_1, X_2)^{-1} = u(Y_1, Y_2)$ for some words Y_1, Y_2. Then the subgroup $\langle Y_1, Y_2\rangle$ of F_m is non-cyclic, for otherwise the equality $u(X_1, X_2)^{-1} = u(Y_1, Y_2)$ would yield that $u(X_1, X_2) = u(Y_1, Y_2)^{-1} = u(Y_1^{-1}, Y_2^{-1})$, contrary to Lemma 4. From here on, follow the proof of Lemma 3 to arrive at a contradiction. $\square$

Lemma 7. *If the subgroup $\langle X_1, X_2\rangle$ of F_m is non-cyclic, then, for each $i = 1, 2$, the subgroup $\langle X_i, u(X_1, X_2)\rangle$ of F_m is also non-cyclic.*

Proof. Suppose on the contrary that $\langle X_i, u(X_1, X_2)\rangle$ is cyclic for some $i = 1, 2$. Then by Lemma 4 we have

$$X_i = u(X_1, X_2)^{\ell} \tag{2.1}$$

for some non-zero integer ℓ. Let U be a cyclically reduced word that is conjugate to $u(X_1, X_2)$, and let C be a simple word such that $[X_1^{48}, X_2^{40}]$ is conjugate to C. By $\overline{X}$, we denote a cyclically reduced word that is conjugate to a word X in F_m. Then by the same arguments as in [2, Lemma 1],

$$\max(|\overline{X_1}|, |\overline{X_2}|) < 6|C|,$$

and also by the same arguments as in [2, Lemma 2], U has the form

$$U \equiv Q_1 S_1 Q_2 S_2 \cdots Q_8 S_8,$$

where Q_i are C-periodic words with $(i \cdot 100 - 14)|C| < |Q_i| \leq i \cdot 100|C|$, $0 \leq |S_i| < 6|C|$ and $3488|C| < |U| < 3648|C|$. This yields that

$$|\overline{X_i}| < 6|C| < 3488|C| < |U| = |\overline{u(X_1, X_2)}| \leq |\overline{u(X_1, X_2)^{\ell}}|,$$

which contradicts equality (2.1). $\square$

In the following lemma which will be useful in Sections 4–6, the word $v_n(x_1, \dots, x_n)$ with $n \geq 3$ is defined by (1.3)–(1.4).

Lemma 8. *If both $v_2(X_1, X_2)$ and $v_n(Y_1, \ldots, Y_n)$ with $n \geq 3$ are neither equal to the empty word nor proper powers, then the subgroup $\langle v_2(X_1, X_2), v_n(Y_1, \ldots, Y_n)\rangle$ of F_m is non-cyclic.*

Proof. Suppose on the contrary that $\langle v_2(X_1, X_2), v_n(Y_1, \ldots, Y_n)\rangle$ is cyclic. It then follows from the hypothesis of the lemma that

$$\text{either} \quad v_2(X_1, X_2) = v_n(Y_1, \ldots, Y_n) \quad \text{or} \quad v_2(X_1, X_2)^{-1} = v_n(Y_1, \ldots, Y_n).$$

This implies, by definition (1.3)–(1.4) of $v_n(x_1, \ldots, x_n)$, the existence of words Z_1, Z_2 in F_m such that

$$\text{either} \quad v_2(X_1, X_2) = u(Z_1, Z_2) \quad \text{or} \quad v_2(X_1, X_2)^{-1} = u(Z_1, Z_2).$$

But since a similar argument to that in Lemma 3 shows that the latter equality cannot hold, the former must hold. From here on, follow the arguments in [2, Lemma 4] to obtain that

$$B^{-\alpha}\tilde{X}_1 B^{\alpha} = \tilde{Z}_1^6 \quad \text{and} \quad B^{-\alpha}\tilde{X}_1^{-1} B^{\alpha} = \tilde{Z}_1^6,$$

where $\tilde{X}_i$, $\tilde{Z}_i$ are conjugates of X_i, Z_i, respectively, and B is a simple word such that $B = [\tilde{X}_1^{8}, \tilde{X}_2^{8}]$. This yields $\tilde{X}_1^{2} = 1$, i.e., $\tilde{X}_1 = 1$, contrary to the hypothesis $v_2(X_1, X_2) \neq 1$. □

3. The case $n = 3$

In this section, we prove that $v_3(x_1, x_2, x_3)$ is a C-test word with the additional property that $v_3(X_1, X_2, X_3) = 1$ if and only if the subgroup $\langle X_1, X_2, X_3\rangle$ of F_m is cyclic. We begin with lemmas that play crucial roles in proving this assertion.

Lemma 9. *If $u\big(v_2(X_1, X_2), v_2(Y_1, Y_2)\big) = 1$, then $v_2(X_1, X_2) = 1$ and $v_2(Y_1, Y_2) = 1$.*

Proof. The hypothesis of the lemma implies by Lemma 4 that $v_2(X_1, X_2)^6 = v_2(Y_1, Y_2)^{-5}$; hence if one of $v_2(X_1, X_2)$ and $v_2(Y_1, Y_2)$ is equal to the empty word, then so is the other. So assume that $v_2(X_1, X_2) \neq 1$ and $v_2(Y_1, Y_2) \neq 1$. Notice that the equality $v_2(X_1, X_2)^6 = v_2(Y_1, Y_2)^{-5}$ implies that $\langle v_2(X_1, X_2), v_2(Y_1, Y_2)\rangle$ is cyclic. Hence, in view of Lemmas 1 and 3, we have $v_2(X_1, X_2) =$

$v_2(Y_1, Y_2)$. This together with $v_2(X_1, X_2)^6 = v_2(Y_1, Y_2)^{-5}$ yields that $v_2(X_1, X_2) = v_2(Y_1, Y_2) = 1$, contrary to our assumption. □

Lemma 10. *Suppose that the subgroup $\langle X_1, X_2, X_3\rangle$ of F_m is non-cyclic. Then $v_3(X_1, X_2, X_3) \neq 1$. Furthermore, either $v_3(X_1, X_2, X_3)$ is not a proper power or it has one of the following three forms:*

$$(A1)\ v_3(X_1, X_2, X_3) = v_2(X_2, X_3)^{960} \quad \text{and} \quad X_1 = 1;$$

$$(A2)\ v_3(X_1, X_2, X_3) = v_2(X_3, X_1)^{400} \quad \text{and} \quad X_2 = 1;$$

$$(A3)\ v_3(X_1, X_2, X_3) = v_2(X_1, X_2)^{576} \quad \text{and} \quad X_3 = 1.$$

Remark. *In view of Lemma 1, $v_2(X_2, X_3)$, $v_2(X_3, X_1)$, $v_2(X_1, X_2)$ in (A1), (A2), (A3), respectively, are neither equal to the empty word nor proper powers.*

Proof. Recall from (1.3) that

$$v_3(X_1, X_2, X_3) = u\Big(u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big),\, u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big)\Big).$$

In the case where the subgroup $\langle u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big),\, u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big)\rangle$ of F_m is non-cyclic, the assertion that $v_3(X_1, X_2, X_3)$ is neither equal to the empty word nor a proper power, as desired, follows immediately from Lemma 4. So we only need to consider the case where

(3.1) the subgroup $\langle u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big),\, u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big)\rangle$ is cyclic.

Here, if $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) = u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) = 1$, then Lemma 9 implies that $v_2(X_1, X_2) = v_2(X_2, X_3) = v_2(X_3, X_1) = 1$; hence, by Lemma 1, $\langle X_1, X_2\rangle$, $\langle X_2, X_3\rangle$ and $\langle X_3, X_1\rangle$ are all cyclic. This yields that $\langle X_1, X_2, X_3\rangle$ is cyclic, contrary to the hypothesis of the lemma. Thus, at least one of the words $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big)$ and $u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big)$ has to be not equal to the empty word. We divide this situation into three cases.

Case I. $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) \neq 1$ *and* $u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) = 1$.

It follows from $u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) = 1$ and Lemma 9 that $v_2(X_2, X_3) = v_2(X_3, X_1) = 1$; so, by Lemma 1, $\langle X_2, X_3\rangle$ and $\langle X_3, X_1\rangle$ are cyclic. Since $\langle X_1, X_2, X_3\rangle$ is non-cyclic, X_3 must be equal to the empty word; hence we have

$$\begin{aligned} v_3(X_1, X_2, X_3) &= u\Big(u\big(v_2(X_1, X_2), 1\big),\ 1\Big) = u\big(v_2(X_1, X_2)^{24}, 1\big) \\ &= v_2(X_1, X_2)^{24\cdot 24} = v_2(X_1, X_2)^{576}. \end{aligned}$$

Therefore $v_3(X_1, X_2, X_3)$ has form $(A3)$ in this case.

Case II. $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) = 1$ *and* $u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) \neq 1$.

Since $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) = 1$, we have, by Lemmas 1 and 9, that $\langle X_1, X_2\rangle$ and $\langle X_2, X_3\rangle$ are cyclic, so that $X_2 = 1$; thus

$$\begin{aligned} v_3(X_1, X_2, X_3) &= u\Big(1,\ u\big(1, v_2(X_3, X_1)\big)\Big) = u\big(1, v_2(X_3, X_1)^{20}\big) \\ &= v_2(X_3, X_1)^{20\cdot 20} = v_2(X_3, X_1)^{400}. \end{aligned}$$

Therefore $v_3(X_1, X_2, X_3)$ has form $(A2)$ in this case.

Case III. $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) \neq 1$ *and* $u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) \neq 1$.

In this case, we want to prove:

Claim. *This case is reduced to the following two cases:*

(i) $\langle v_2(X_1, X_2), v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ *is cyclic*;

(ii) *both* $\langle v_2(X_1, X_2), v_2(X_2, X_3)\rangle$ *and* $\langle v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ *are non-cyclic.*

Proof of the Claim. Assuming at least one of $\langle v_2(X_1, X_2), v_2(X_2, X_3)\rangle$ and $\langle v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ is cyclic, we want to show that Case (i) occurs. Let us say that $\langle v_2(X_1, X_2), v_2(X_2, X_3)\rangle$ is cyclic (the case where $\langle v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ is cyclic is analogous). If $v_2(X_2, X_3) = 1$, then $u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) = v_2(X_1, X_2)^{24}$ and $u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) = v_2(X_3, X_1)^{20}$. It then

follows from (3.1) that $\langle v_2(X_1, X_2), v_2(X_3, X_1)\rangle$ is cyclic, which means that Case (i) occurs. Now let $v_2(X_2, X_3) \neq 1$. Then since $\langle v_2(X_1, X_2), v_2(X_2, X_3)\rangle$ is cyclic, (3.1) yields by Lemma 7 that $\langle v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ is also cyclic; hence $\langle v_2(X_1, X_2), v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ is cyclic, that is, Case (i) occurs as well. □

Case (i) is again divided into subcases according to the number of non-empty words among $v_2(X_1, X_2)$, $v_2(X_2, X_3)$ and $v_2(X_3, X_1)$. Here, we note that if there exists only one non-empty word, then it has to be $v_2(X_2, X_3)$, for otherwise we would have a contradiction to the hypothesis of Case III. Therefore, Case III is decomposed into the following six subcases.

Case III.1. *$v_2(X_1, X_2) = v_2(X_3, X_1) = 1$ and $v_2(X_2, X_3) \neq 1$.*

In this case, it follows from Lemma 1 that $\langle X_1, X_2\rangle$ and $\langle X_3, X_1\rangle$ are cyclic, so that $X_1 = 1$; hence we have

$$\begin{aligned} v_3(X_1, X_2, X_3) &= u\Big(u\big(1, v_2(X_2, X_3)\big),\, u\big(v_2(X_2, X_3), 1\big)\Big) = u\big(v_2(X_2, X_3)^{20}, v_2(X_2, X_3)^{24}\big) \\ &= v_2(X_2, X_3)^{20\cdot 24+24\cdot 20} = v_2(X_2, X_3)^{960}. \end{aligned}$$

Thus, $v_3(X_1, X_2, X_3)$ has form $(A1)$.

Case III.2. *$v_2(X_1, X_2) = 1$ and $\langle 1 \neq v_2(X_2, X_3), 1 \neq v_2(X_3, X_1)\rangle$ is cyclic.*

Since $v_2(X_1, X_2) = 1$, we have by Lemma 1 that $\langle X_1, X_2\rangle$ is cyclic. Also since $\langle 1 \neq v_2(X_2, X_3), 1 \neq v_2(X_3, X_1)\rangle$ is cyclic, we have $1 \neq v_2(X_2, X_3) = v_2(X_3, X_1)$ by Lemmas 1 and 3. Apply Lemma 2 to this equality: there exists a word $S \in F_m$ such that

$$X_2 = SX_3S^{-1} \quad \text{and} \quad X_3 = SX_1S^{-1},$$

which yields that $S^{-1}X_2S = SX_1S^{-1}$, so that $X_2 = S^2X_1S^{-2}$. It then follows from $\langle X_1, X_2\rangle$ being cyclic that $\langle S, X_1, X_2\rangle$ is cyclic. This together with the equality $X_3 = SX_1S^{-1}$ implies that $\langle X_1, X_2, X_3\rangle$ is cyclic, contrary to the hypothesis of the lemma. Therefore this case cannot occur.

Case III.3. *$v_2(X_2, X_3) = 1$ and $\langle 1 \neq v_2(X_1, X_2), 1 \neq v_2(X_3, X_1)\rangle$ is cyclic.*

Repeat a similar argument to that in Case III.2 to conclude that this case cannot occur.

Case III.4. *$v_2(X_3, X_1) = 1$ and $\langle 1 \neq v_2(X_1, X_2), 1 \neq v_2(X_2, X_3)\rangle$ is cyclic.*

Also repeat a similar argument to that in Case III.2 to conclude that this case cannot occur.

Case III.5. *$\langle 1 \neq v_2(X_1, X_2), 1 \neq v_2(X_2, X_3), 1 \neq v_2(X_3, X_1)\rangle$ is cyclic.*

In this case, it follows from Lemmas 1 and 3 that

$$1 \neq v_2(X_1, X_2) = v_2(X_2, X_3) = v_2(X_3, X_1). \tag{3.2}$$

Applying Lemma 2 to these equalities, we have the existence of words T_1 and T_2 in F_m such that

$$\begin{aligned} X_1 = T_1 X_2 T_1^{-1}, \quad X_2 = T_1 X_3 T_1^{-1}; \\ X_2 = T_2 X_3 T_2^{-1}, \quad X_3 = T_2 X_1 T_2^{-1}. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) yields that $\langle T_1^{-1}T_2, X_3\rangle$, $\langle T_1, v_2(X_2, X_3)\rangle$ and $\langle T_2, v_2(X_3, X_1)\rangle$ are all cyclic. At this point, we apply Ivanov's argument (see [2, pp. 403–404]) to obtain the following:

Claim (Ivanov). *$T_1 = T_2$.*

Proof of the Claim. Suppose on the contrary that $T_1 \neq T_2$. It then follows from $\langle T_1^{-1}T_2, X_3\rangle$ being cyclic that

$$X_3^{\ell_1} = (T_1^{-1}T_2)^{\ell_2} \tag{3.4}$$

with nonzero integers ℓ_1 and ℓ_2. It also follows from $\langle T_1, v_2(X_2, X_3)\rangle$ and $\langle T_2, v_2(X_3, X_1)\rangle$ being cyclic that

$$T_1 = v_2(X_2, X_3)^{\ell_3} \quad \text{and} \quad T_2 = v_2(X_3, X_1)^{\ell_4},$$

with integers l_3 and l_4 at least one of which is non-zero, so that

$$T_1^{-1}T_2 = v_2(X_2, X_3)^{-\ell_3} v_2(X_3, X_1)^{\ell_4}.$$

Hence, by (3.4),

$$X_3^{\ell_1} = [v_2(X_2, X_3)^{-\ell_3} v_2(X_3, X_1)^{\ell_4}]^{\ell_2}. \tag{3.5}$$

Note, by definition (1.2) of $v_2(x_1, x_2)$, that the right hand side of equality (3.5) belongs to the subgroup $[F_m, \mathcal{N}_3]$, where $\mathcal{N}_3$ is the normal closure in F_m of the word X_3. So inside the relation module $\hat{\mathcal{N}}_3 = \mathcal{N}_3/[\mathcal{N}_3, \mathcal{N}_3]$ of the one-relator group

$$G = \langle x_1, \ldots, x_m \| X_3 \rangle,$$

equality (3.5) can be expressed as

$$(\ell_1 - P) \cdot \hat{X}_3 = 0,$$

where P is an element of the augmentation ideal of the group ring $\mathbb{Z}(G)$ of G over the integers and $\hat{X}_3$ is the canonical generator of the relation module $\hat{\mathcal{N}}_3$ of G. By Lyndon's result on the relation module $\mathcal{R}$ of a one-relator group $\langle x_1, \ldots, x_m \| R \rangle$ (see [4, 5]) which says that if $Q \cdot \hat{R} = 0$ in $\mathcal{R}$ then Q is an element of the augmentation ideal of $\mathbb{Z}(\langle x_1, \ldots, x_m \| R \rangle)$, we must have $\ell_1 = 0$. This contradiction to the fact $\ell_1 \neq 0$ completes the proof of the claim. $\square$

If $T_1 = T_2 = 1$, then equalities (3.3) yield that $X_1 = X_2 = X_3$, contrary to the hypothesis of the lemma. If $T_1 = T_2 \neq 1$, then we derive from equalities (3.3) that

$$X_1 = T_1^3 X_1 T_1^{-3}, \quad X_2 = T_1^3 X_2 T_1^{-3} \quad \text{and} \quad X_3 = T_1^3 X_3 T_1^{-3},$$

so that $\langle T_1, X_1 \rangle$, $\langle T_1, X_2 \rangle$ and $\langle T_1, X_3 \rangle$ are all cyclic; therefore, $\langle X_1, X_2, X_3 \rangle$ is cyclic. A contradiction implies that this case cannot occur.

Case III.6. *Both $\langle v_2(X_1, X_2), v_2(X_2, X_3) \rangle$ and $\langle v_2(X_2, X_3), v_2(X_3\, X_1) \rangle$ are non-cyclic.*

In this case, in view of (3.1) and Lemmas 4 and 6, we have that

$$u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) = u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big),$$

so by Lemma 5 that there exists a word $T \in F_m$ such that

$$1 \neq v_2(X_1, X_2) = T v_2(X_2, X_3) T^{-1} \text{ and } 1 \neq v_2(X_2, X_3) = T v_2(X_3, X_1) T^{-1}. \tag{3.6}$$

Apply Lemma 2 to these equalities: there exist words U_1 and U_2 in F_m such that

$$\begin{aligned} X_1 &= U_1 X_2 U_1^{-1}, \quad X_2 = U_1 X_3 U_1^{-1}; \\ X_2 &= U_2 X_3 U_2^{-1}, \quad X_3 = U_2 X_1 U_2^{-1}. \end{aligned} \tag{3.7}$$

Combining the equalities in (3.6) and (3.7), we deduce that $\langle U_1^{-1} U_2, X_3 \rangle$, $\langle T^{-1} U_1, v_2(X_2, X_3) \rangle$ and $\langle T^{-1} U_2, v_2(X_3, X_1) \rangle$ are cyclic. Here, apply Ivanov's argument used in Case III.5 to get $U_1 = U_2$. Then reasoning as in Case III.5, we conclude that this case cannot occur.

The proof of Lemma 10 is complete. □

Now we are ready to prove the Theorem for the case $n = 3$.

Proof of the Theorem ($n = 3$). The additional property that $v_3(X_1, X_2, X_3) = 1$ if and only if the subgroup $\langle X_1, X_2, X_3 \rangle$ of F_m is cyclic follows immediately from definition (1.3) of $v_3(x_1, x_2, x_3)$ and Lemma 10. So we only need to prove that $v_3(x_1, x_2, x_3)$ is a C-test word, that is, supposing $1 \neq v_3(X_1, X_2, X_3) = v_3(Y_1, Y_2, Y_3)$, we want to prove the existence of a word $Z \in F_m$ such that

$$Y_i = Z X_i Z^{-1} \quad \text{for all} \quad i = 1,\, 2,\, 3.$$

We begin by dividing into two cases according to whether $v_3(X_1, X_2, X_3)$ is a proper power or not.

Case I. $v_3(X_1, X_2, X_3)$ *is a proper power.*

Applying Lemma 10 to $v_3(X_1, X_2, X_3)$ and $v_3(Y_1, Y_2, Y_3)$, we have: $v_3(X_1, X_2, X_3)$ has one of three types $(A1)$, $(A2)$ and $(A3)$; besides, by the equality $v_3(X_1, X_2, X_3) = v_3(Y_1, Y_2, Y_3)$, $v_3(Y_1, Y_2, Y_3)$ has the same type as $v_3(X_1, X_2, X_3)$, because the exponents in $(A1)$, $(A2)$ and $(A3)$ are all distinct. This gives us only three possibilities $(A1)$&$(A1)$, $(A2)$&$(A2)$ and $(A3)$&$(A3)$ for the types of $v_3(X_1, X_2, X_3)$ & $v_3(Y_1, Y_2, Y_3)$.

If $v_3(X_1, X_2, X_3) \,\&\, v_3(Y_1, Y_2, Y_3)$ is of type $(A1)\&(A1)$ ($(A2)\&(A2)$ or $(A3)\&(A3)$ is similar), then

$$X_1 = Y_1 = 1 \quad \text{and} \quad 1 \neq v_2(X_2, X_3)^{960} = v_2(Y_2, Y_3)^{960}.$$

Applying Lemma 2 to the equality $1 \neq v_2(X_2, X_3) = v_2(Y_2, Y_3)$, we have that two 2-tuples (X_2, X_3) and (Y_2, Y_3) are conjugate in F_m, which together with $X_1 = Y_1 = 1$ yields that two 3-tuples (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) are conjugate in F_m, as desired.

Case II. *$v_3(X_1, X_2, X_3)$ is not a proper power.*

In this case, it follows from Lemma 4 that

$$\text{(3.8)} \quad \text{the subgroup } \langle u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big),\ u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big)\rangle \text{ of } F_m \text{ is non-cyclic.}$$

This enables us to apply Lemma 5 to the equality $v_3(X_1, X_2, X_3) = v_3(Y_1, Y_2, Y_3)$: for some word $S \in F_m$, we have

$$\begin{aligned} &1 \neq u\big(v_2(X_1, X_2), v_2(X_2, X_3)\big) = Su\big(v_2(Y_1, Y_2), v_2(Y_2, Y_3)\big)S^{-1}, \\ &1 \neq u\big(v_2(X_2, X_3), v_2(X_3, X_1)\big) = Su\big(v_2(Y_2, Y_3), v_2(Y_3, Y_1)\big)S^{-1}. \end{aligned} \tag{3.9}$$

Here, we consider two subcases.

Case II.1. *One of $\langle v_2(X_1, X_2), v_2(X_2, X_3)\rangle$ and $\langle v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ is non-cyclic.*

Let us say that $\langle v_2(X_1, X_2), v_2(X_2, X_3)\rangle$ is non-cyclic (the case where $\langle v_2(X_2, X_3), v_2(X_3, X_1)\rangle$ is non-cyclic is analogous). Then, by Lemma 5, the first equality of (3.9) implies the existence of a word $W \in F_m$ such that

$$1 \neq v_2(X_1, X_2) = Wv_2(Y_1, Y_2)W^{-1} \quad \text{and} \quad 1 \neq v_2(X_2, X_3) = Wv_2(Y_2, Y_3)W^{-1}. \tag{3.10}$$

Apply Lemma 2 to these equalities: there exist words T_1 and T_2 in F_m such that

$$\begin{aligned} X_1 &= T_1Y_1T_1^{-1}, \quad X_2 = T_1Y_2T_1^{-1}; \\ X_2 &= T_2Y_2T_2^{-1}, \quad X_3 = T_2Y_3T_2^{-1}. \end{aligned} \tag{3.11}$$

Now applying Ivanov's argument introduced in Case III.5 of Lemma 10 to equalities (3.10)–(3.11), we get $T_1 = T_2$, by which equalities (3.11) yield the desired result.

Case II.2. *Both* $\langle v_2(X_1,X_2), v_2(X_2,X_3)\rangle$ *and* $\langle v_2(X_2,X_3), v_2(X_3,X_1)\rangle$ *are cyclic.*

In this case, if $v_2(X_2,X_3) \neq 1$, then the subgroup $\langle v_2(X_1,X_2), v_2(X_2,X_3), v_2(X_3,X_1)\rangle$ would be cyclic, contrary to (3.8). So $v_2(X_2,X_3)$ must be equal to the empty word. On the other hand, equalities (3.9) imply by Lemma 4 that both $\langle v_2(Y_1,Y_2), v_2(Y_2,Y_3)\rangle$ and $\langle v_2(Y_2,Y_3), v_2(Y_3,Y_1)\rangle$ are also cyclic. Then, for the same reason as $v_2(X_2,X_3)$, $v_2(Y_2,Y_3)$ also has to be equal to the empty word.

Thus, it follows from (3.9) that

$$1 \neq v_2(X_1,X_2)^{24} = Sv_2(Y_1,Y_2)^{24}S^{-1} \quad \text{and} \quad 1 \neq v_2(X_3,X_1)^{20} = Sv_2(Y_3,Y_1)^{20}S^{-1},$$

that is,

$$1 \neq v_2(X_1,X_2) = Sv_2(Y_1,Y_2)S^{-1} \quad \text{and} \quad 1 \neq v_2(X_3,X_1) = Sv_2(Y_3,Y_1)S^{-1},$$

which is a similar situation to (3.10). So from here on, we can follow the proof of Case II.1 to obtain the desired result.

The proof of the Theorem for the case $n = 3$ is complete. □

4. The base $n = 4$ of simultaneous induction

In this section, we prove the base step $n = 4$ of simultaneous induction which we use in Lemmas 11–13 and the Theorem.

Lemma 11 (n=4). *If both* $v_2(X_1,X_2)$ *and* $v_3(Y_1,Y_2,Y_3)$ *are neither equal to the empty word nor proper powers, then* $\langle v_2(X_1,X_2), v_3(Y_1,Y_2,Y_3)\rangle$ *is non-cyclic.*

Proof. This is a special case of Lemma 8. □

Lemma 12 (n=4). *If* $u\big(v_3(X_1,X_2,X_3), v_3(Y_1,Y_2,Y_3)\big) = 1$, $v_3(X_1,X_2,X_3) = v_3(Y_1,Y_2,Y_3) = 1$.

Proof. By Lemma 4, the hypothesis of the lemma implies that

$$v_3(X_1,X_2,X_3)^6 = v_3(Y_1,Y_2,Y_3)^{-5}, \tag{4.1}$$

so that if one of $v_3(X_1, X_2, X_3)$ and $v_3(Y_1, Y_2, Y_3)$ is equal to the empty word, then so is the other. Hence assume that $v_3(X_1, X_2, X_3) \neq 1$ and $v_3(Y_1, Y_2, Y_3) \neq 1$.

If one of $v_3(X_1, X_2, X_3)$ and $v_3(Y_1, Y_2, Y_3)$ is a proper power, then so is the other by (4.1), because 6 and 5 are relatively prime. Hence by Lemma 10 $v_3(X_1, X_2, X_3) \,\&\, v_3(Y_1, Y_2, Y_3)$ has nine possible types, and we can easily check that 6 times any of 960, 400 and 576 never equals 5 times any of these, which means that equality (4.1) cannot hold in any case, a contradiction.

If neither $v_3(X_1, X_2, X_3)$ nor $v_3(Y_1, Y_2, Y_3)$ is a proper power, then by Lemma 6 we have $v_3(X_1, X_2, X_3) = v_3(Y_1, Y_2, Y_3)$, because (4.1) implies that $\langle v_3(X_1, X_2, X_3), v_3(Y_1, Y_2, Y_3)\rangle$ is cyclic. This equality together with (4.1) yields that $v_3(X_1, X_2, X_3) = v_3(Y_1, Y_2, Y_3) = 1$, contrary to our assumption. This completes the proof. $\square$

Lemma 13 (n=4). *Suppose that $\langle X_1, X_2, X_3, X_4\rangle$ is non-cyclic. Then $v_4(X_1, X_2, X_3, X_4) \neq 1$. Furthermore, either $v_4(X_1, X_2, X_3, X_4)$ is not a proper power or it has one of the following four forms:*

$$(B1)\ v_4(X_1, X_2, X_3, X_4) = v_2(X_4, X_3)^{960 \cdot 400} \quad \text{and} \quad X_1 = X_2 = 1;$$

$$(B2)\ v_4(X_1, X_2, X_3, X_4) = v_2(X_1, X_4)^{400^2} \quad \text{and} \quad X_2 = X_3 = 1;$$

$$(B3)\ v_4(X_1, X_2, X_3, X_4) = v_2(X_3, X_1)^{576 \cdot 400} \quad \text{and} \quad X_2 = X_4 = 1;$$

$$(B4)\ v_4(X_1, X_2, X_3, X_4) = v_3(X_1, X_2, X_3)^{1936} \quad \text{and} \quad X_1 = X_3 = X_4 \neq 1.$$

Remark. *In view of Lemmas 1 and 10, $v_2(X_4, X_3)$, $v_2(X_1, X_4)$, $v_2(X_3, X_1)$ and $v_3(X_1, X_2, X_3)$ in $(B1)$, $(B2)$, $(B3)$ and $(B4)$, respectively, are neither equal to the empty word nor proper powers.*

Proof. Recall from (1.5) that

$$v_4(X_1, X_2, X_3, X_4) = u\Big(u\big(v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\big),\ u\big(v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\big)\Big).$$

If $\langle u\big(v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\big),\ u\big(v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\big)\rangle$ is non-cyclic, then the assertion that $v_4(X_1, X_2, X_3, X_4)$ is neither equal to the empty word nor a proper power, as desired,

follows directly from Lemma 4. So we only need to consider the case where

$$\langle u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big),\ u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\big)\rangle \text{ is cyclic.} \tag{4.2}$$

Here, in order to avoid a contradiction to the hypothesis that $\langle X_1,X_2,X_3,X_4\rangle$ is non-cyclic, at least one of $u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big)$ and $u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\big)$ has to be not equal to the empty word. So we have three cases to consider.

Case I. $u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big)\neq 1$ *and* $u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\big)=1$.

In this case, we have, by Lemma 12 ($n=4$) and the Theorem ($n=3$), that both $\langle X_3,X_2,X_4\rangle$ and $\langle X_4,X_2,X_1\rangle$ are cyclic. Since $\langle X_1,X_2,X_3,X_4\rangle$ is non-cyclic, X_2 and X_4 must be equal to the empty word; hence we have

$$\begin{aligned} v_4(X_1,X_2,X_3,X_4) &= u\Big(u\big(v_3(X_1,X_2,X_3),1\big),\ 1\Big) \\ &= u\big(v_3(X_1,X_2,X_3)^{24},1\big) = v_3(X_1,X_2,X_3)^{24\cdot 24} \\ &= v_2(X_3,X_1)^{576\cdot 400} \quad \text{by form } (A2). \end{aligned}$$

Therefore, $v_4(X_1,X_2,X_3,X_4)$ has form $(B3)$ in this case.

Case II. $u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big)=1$ *and* $u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\big)\neq 1$.

In this case, by Lemma 12 ($n=4$) and the Theorem ($n=3$), we have that both $\langle X_1,X_2,X_3\rangle$ and $\langle X_3,X_2,X_4\rangle$ are cyclic, so that $X_2=X_3=1$; thus

$$\begin{aligned} v_4(X_1,X_2,X_3,X_4) &= u\Big(1,\ u\big(1,v_3(X_4,X_2,X_1)\big)\Big) \\ &= u\big(1,v_3(X_4,X_2,X_1)^{20}\big) = v_3(X_4,X_2,X_1)^{20\cdot 20} \\ &= v_2(X_1,X_4)^{400\cdot 400} \quad \text{by form } (A2). \end{aligned}$$

Therefore, $v_4(X_1,X_2,X_3,X_4)$ has form $(B2)$ in this case.

Case III. $u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big)\neq 1$ *and* $u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\big)\neq 1$.

By reasoning as in Case III of Lemma 10, we break this case into the following six subcases.

Case III.1. $v_3(X_1, X_2, X_3) = v_3(X_4, X_2, X_1) = 1$ *and* $v_3(X_3, X_2, X_4) \neq 1$.

It follows from the Theorem ($n = 3$) that $\langle X_1, X_2, X_3\rangle$ and $\langle X_4, X_2, X_1\rangle$ are cyclic, so that $X_1 = X_2 = 1$; hence we have

$$\begin{aligned} v_4(X_1, X_2, X_3, X_4) &= u\Big(u\big(1, v_3(X_3, X_2, X_4)\big),\, u\big(v_3(X_3, X_2, X_4), 1\big)\Big) \\ &= u\big(v_3(X_3, X_2, X_4)^{20}, v_3(X_3, X_2, X_4)^{24}\big) = v_3(X_3, X_2, X_4)^{20\cdot 24+24\cdot 20} \\ &= v_2(X_4, X_3)^{960\cdot 400} \quad \text{by form } (A2). \end{aligned}$$

Thus, $v_4(X_1, X_2, X_3, X_4)$ has form $(B1)$.

Case III.2. $v_3(X_1, X_2, X_3) = 1$ *and* $\langle 1 \neq v_3(X_3, X_2, X_4), 1 \neq v_3(X_4, X_2, X_1)\rangle$ *is cyclic.*

Since $v_3(X_1, X_2, X_3) = 1$, we have, by the Theorem ($n = 3$), that

$$\langle X_1, X_2, X_3\rangle \ \text{ is cyclic.} \tag{4.3}$$

Also since $\langle 1 \neq v_3(X_3, X_2, X_4), 1 \neq v_3(X_4, X_2, X_1)\rangle$ is cyclic, in view of Lemmas 10 and 11 ($n = 4$), this case is reduced to the following two cases:

(i) both $v_3(X_3, X_2, X_4)$ and $v_3(X_4, X_2, X_1)$ are proper powers;

(ii) neither $v_3(X_3, X_2, X_4)$ nor $v_3(X_4, X_2, X_1)$ is a proper power.

Case (i) is divided further into subcases according to types of $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$ by Lemma 10. Of nine possible types of $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$, $(A3)\&(A1)$, $(A3)\&(A2)$ and $(A3)\&(A3)$ cannot occur, for if $v_3(X_3, X_2, X_4)$ were of type $(A3)$, then $X_4 = 1$, which together with (4.3) yields a contradiction to the hypothesis of the lemma. Also, $(A1)\&(A1)$ and $(A2)\&(A1)$ cannot occur, for if $v_3(X_4, X_2, X_1)$ were of type $(A1)$, then $X_4 = 1$, again a contradiction. Moreover, $(A1)\&(A2)$ cannot occur, for this implies that $X_3 = X_2 = 1$, so that $v_3(X_3, X_2, X_4) = 1$, a contradiction. Also, $(A2)\&(A3)$ cannot occur, for this implies that $X_2 = X_1 = 1$, so that $v_3(X_4, X_2, X_1) = 1$, a contradiction as well.

For this reason, in Case (i), we only need to consider $(A1)\&(A3)$ and $(A2)\&(A2)$ for the types of $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$. Therefore, Case III.2 is decomposed into the following three subcases.

Case III.2.1. $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$ *is of type* $(A1)\&(A3)$.

In this case, it follows from Lemma 10 that $X_3 = X_1 = 1$, $v_3(X_3, X_2, X_4) = v_2(X_2, X_4)^{960}$ and $v_3(X_4, X_2, X_1) = v_2(X_4, X_2)^{576}$. Since $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ is cyclic by the hypothesis of Case III.2, we have that $\langle v_2(X_2, X_4), v_2(X_4, X_2)\rangle$ is cyclic, so that $1 \neq v_2(X_2, X_4) = v_2(X_4, X_2)$ by Lemmas 1 and 3. Apply Lemma 2 to this equality: there is a word $S \in F_m$ such that

$$X_2 = SX_4S^{-1} \quad \text{and} \quad X_4 = SX_2S^{-1}. \tag{4.4}$$

If $S = 1$, then from (4.4) we have $X_2 = X_4$, which together with (4.3) yields a contradiction to the hypothesis of the lemma. Now let $S \neq 1$. We derive from (4.4) that

$$X_2 = S^2X_2S^{-2} \quad \text{and} \quad X_4 = S^2X_4S^{-2},$$

so that $\langle S, X_2\rangle$ and $\langle S, X_4\rangle$ are cyclic; thus $\langle X_2, X_4\rangle$ is cyclic. This together with (4.3) yields a contradiction as well (because $X_2 \neq 1$). Therefore, we conclude that this case cannot occur.

Case III.2.2. $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$ *is of type* $(A2)\&(A2)$.

It follows from Lemma 10 that $X_2 = 1$, $v_3(X_3, X_2, X_4) = v_2(X_4, X_3)^{400}$ and $v_3(X_4, X_2, X_1) = v_2(X_1, X_4)^{400}$. Since $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ is cyclic, $\langle v_2(X_4, X_3), v_2(X_1, X_4)\rangle$ is cyclic, hence, by Lemmas 1 and 3, $1 \neq v_2(X_4, X_3) = v_2(X_1, X_4)$. Then by Lemma 2 there exists a word $U \in F_m$ such that

$$X_4 = UX_1U^{-1} \quad \text{and} \quad X_3 = UX_4U^{-1}. \tag{4.5}$$

If $U = 1$, then it follows from (4.5) that $X_1 = X_4 = X_3$, which together with (4.3) yields a contradiction to the hypothesis of the lemma. Now let $U \neq 1$. We have from (4.5) that $UX_1U^{-1} =$

$U^{-1}X_3U$. This equality implies by (4.3) that $\langle U, X_1, X_2, X_3\rangle$ is cyclic; thus, by the first equality of (4.5), we have that $\langle X_1, X_2, X_3, X_4\rangle$ is cyclic. A contradiction implies that this case cannot occur.

Case III.2.3. *Neither $v_3(X_3, X_2, X_4)$ nor $v_3(X_4, X_2, X_1)$ is a proper power.*

Since $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ is cyclic, we have, by Lemma 6, that

$$1 \neq v_3(X_3, X_2, X_4) = v_3(X_4, X_2, X_1).$$

Apply the Theorem ($n = 3$) to this equality: there is a word $T \in F_m$ such that

$$X_3 = TX_4T^{-1}, \quad X_2 = TX_2T^{-1} \quad \text{and} \quad X_4 = TX_1T^{-1}. \tag{4.6}$$

The second equality of (4.6) implies that $\langle T, X_2\rangle$ is cyclic; hence, by (4.3), $\langle T, X_1, X_2, X_3\rangle$ is cyclic (because $X_2 \neq 1$). Then by the third equality of (4.6), we have that $\langle X_1, X_2, X_3, X_4\rangle$ is cyclic. A contradiction implies that this case cannot occur.

Case III.3. *$v_3(X_3, X_2, X_4) = 1$ and $\langle 1 \neq v_3(X_1, X_2, X_3), 1 \neq v_3(X_4, X_2, X_1)\rangle$ is cyclic.*

Repeat similar arguments to those in Case III.2 to conclude that this case cannot occur.

Case III.4. *$v_3(X_4, X_2, X_1) = 1$ and $\langle 1 \neq v_3(X_1, X_2, X_3), 1 \neq v_3(X_3, X_2, X_4)\rangle$ is cyclic.*

Also, repeat similar arguments to those in Case III.2 to conclude that this case cannot occur.

Case III.5. *$\langle 1 \neq v_3(X_1, X_2, X_3), 1 \neq v_3(X_3, X_2, X_4), 1 \neq v_3(X_4, X_2, X_1)\rangle$ is cyclic.*

In this case, we want to prove:

Claim. $1 \neq v_3(X_1, X_2, X_3) = v_3(X_3, X_2, X_4) = v_3(X_4, X_2, X_1)$.

Proof of the Claim. If none of these is a proper power, then the assertion follows immediately from Lemma 6. So assume one of these is a proper power. Then, in view of Lemmas 10 and 11 ($n = 4$), the other two also have to be proper powers; thus two of X_1, X_3 and X_4 must be equal to the empty word, unless $X_2 = 1$. However, if two of X_1, X_3 and X_4 were equal to the empty

word, then we would have a contradiction to the non-triviality of $v_3(X_1, X_2, X_3)$, $v_3(X_3, X_2, X_4)$ or $v_3(X_4, X_2, X_1)$. Hence we must have $X_2 = 1$. Then

(4.7)

$$v_3(X_1, X_2, X_3) = v_2(X_3, X_1)^{400},\ v_3(X_3, X_2, X_4) = v_2(X_4, X_3)^{400},\ v_3(X_4, X_2, X_1) = v_2(X_1, X_4)^{400};$$

hence the hypothesis of Case III.5 implies that $\langle v_2(X_3, X_1), v_2(X_4, X_3), v_2(X_1, X_4)\rangle$ is cyclic. It then follows from Lemmas 1 and 3 that $v_2(X_3, X_1) = v_2(X_4, X_3) = v_2(X_1, X_4)$, which together with (4.7) proves the claim. $\square$

Now apply the Theorem ($n = 3$) to the equalities in the Claim: there exist words Z_1 and Z_2 in F_m such that

$$\begin{aligned} &X_1 = Z_1 X_3 Z_1^{-1}, \quad X_2 = Z_1 X_2 Z_1^{-1}, \quad X_3 = Z_1 X_4 Z_1^{-1}; \\ &X_3 = Z_2 X_4 Z_2^{-1}, \quad X_2 = Z_2 X_2 Z_2^{-1}, \quad X_4 = Z_2 X_1 Z_2^{-1}. \end{aligned} \tag{4.8}$$

We deduce from these equalities that $\langle Z_1, X_2\rangle$, $\langle Z_2, X_2\rangle$, $\langle Z_1Z_2^2, Z_1^2Z_2, X_1\rangle$, $\langle Z_1Z_2^2, Z_1^2Z_2, X_3\rangle$ and $\langle Z_1Z_2^2, Z_1^2Z_2, X_4\rangle$ are all cyclic. Here, if either $Z_1Z_2^2 \neq 1$ or $Z_1^2Z_2 \neq 1$, then we would have that $\langle X_1, X_2, X_3, X_4\rangle$ is cyclic, a contradiction. Hence we must have that $Z_1Z_2^2 = Z_1^2Z_2 = 1$, that is, $Z_1 = Z_2 = 1$; thus, by (4.8),

$$X_1 = X_3 = X_4.$$

In addition, it follows from the Claim that

$$\begin{aligned} v_4(X_1, X_2, X_3, X_4) &= u\Big(u\big(v_3(X_1, X_2, X_3), v_3(X_1, X_2, X_3)\big),\ u\big(v_3(X_1, X_2, X_3), v_3(X_1, X_2, X_3)\big)\Big) \\ &= u\big(v_3(X_1, X_2, X_3)^{44}, v_3(X_1, X_2, X_3)^{44}\big) \\ &= v_3(X_1, X_2, X_3)^{44\cdot 24 + 44\cdot 20} \\ &= v_3(X_1, X_2, X_3)^{1936}. \end{aligned}$$

Therefore, in this case, $v_4(X_1, X_2, X_3, X_4)$ has form $(B4)$.

Case III.6. *Both* $\langle v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\rangle$ *and* $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ *are non-cyclic.*

In view of (4.2) and Lemmas 4 and 6, we have that

$$u\big(v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\big) = u\big(v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\big).$$

Then by Lemma 5 applied to this equality, there is a word $W \in F_m$ such that

$$1 \neq v_3(X_1, X_2, X_3) = W v_3(X_3, X_2, X_4) W^{-1} \quad \text{and} \quad 1 \neq v_3(X_3, X_2, X_4) = W v_3(X_4, X_2, X_1) W^{-1}.$$

Applying the Theorem ($n = 3$) to these equalities yields the existence of words V_1 and V_2 in F_m such that

$$X_1 = V_1 X_3 V_1^{-1}, \quad X_2 = V_1 X_2 V_1^{-1}, \quad X_3 = V_1 X_4 V_1^{-1};$$

$$X_3 = V_2 X_4 V_2^{-1}, \quad X_2 = V_2 X_2 V_2^{-1}, \quad X_4 = V_2 X_1 V_2^{-1}.$$

This is the same situation as (4.8); hence, reasoning as in Case III.5, we have $X_1 = X_3 = X_4$. But then $v_3(X_1, X_2, X_3) = v_3(X_3, X_2, X_4) = v_3(X_4, X_2, X_1)$, which yields a contradiction to the hypothesis of Case III.6. Therefore, we conclude that this case cannot occur.

The proof of Lemma 13 ($n = 4$) is now complete. □

Proof of the Theorem ($n = 4$). The additional property that $v_4(X_1, X_2, X_3, X_4) = 1$ if and only if the subgroup $\langle X_1, X_2, X_3, X_4 \rangle$ of F_m is cyclic is immediate from definition (1.5) of $v_4(x_1, x_2, x_3, x_4)$ and Lemma 13 ($n = 4$). Now we want to prove that $v_4(x_1, x_2, x_3, x_4)$ is a C-test word, that is, supposing $1 \neq v_4(X_1, X_2, X_3, X_4) = v_4(Y_1, Y_2, Y_3, Y_4)$, we want to prove the existence of a word Z in F_m such that

$$Y_i = Z X_i Z^{-1} \quad \text{for all} \quad i = 1,\ 2,\ 3,\ 4.$$

We consider two cases corresponding to whether $v_4(X_1, X_2, X_3, X_4)$ is a proper power or not.

Case I. $v_4(X_1, X_2, X_3, X_4)$ *is a proper power.*

Apply Lemma 13 ($n = 4$) to $v_4(X_1, \ldots, X_4)$ and $v_4(Y_1, \ldots, Y_4)$: $v_4(X_1, \ldots, X_4)$ has one of the four types $(B1)$–$(B4)$; besides, by the equality $v_4(X_1, \ldots, X_4) = v_4(Y_1, \ldots, Y_4)$, $v_4(Y_1, \ldots, Y_4)$ has the same type as $v_4(X_1, \ldots, X_4)$, since the exponents in $(B1)$–$(B4)$ are all distinct. This gives us only four possibilities $(B1)\&(B1), \cdots, (B4)\&(B4)$ for the types of $v_4(X_1, \ldots, X_4) \,\&\, v_4(Y_1, \ldots, Y_4)$.

If $v_4(X_1,\dots,X_4)\,\&\,v_4(Y_1,\dots,Y_4)$ is of type $(B1)\&(B1)$ ($(B2)\&(B2)$ or $(B3)\&(B3)$ is analogous), then

$$X_1=X_2=Y_1=Y_2=1 \quad \text{and} \quad 1\neq v_2(X_4,X_3)^{960\cdot 400}=v_2(Y_4,Y_3)^{960\cdot 400}.$$

Applying Lemma 2 to the equality $1\neq v_2(X_4,X_3)=v_2(Y_4,Y_3)$, we have that two 2-tuples (X_4,X_3) and (Y_4,Y_3) are conjugate in F_m, which together with $X_1=X_2=Y_1=Y_2=1$ yields the desired result.

If $v_4(X_1,\dots,X_4)\,\&\,v_4(Y_1,\dots,Y_4)$ is of type $(B4)\&(B4)$, then

$$X_1=X_3=X_4, \quad Y_1=Y_3=Y_4 \ \text{ and } \ 1\neq v_3(X_1,X_2,X_3)^{1936}=v_3(Y_1,Y_2,Y_3)^{1936}.$$

The equality $1\neq v_3(X_1,X_2,X_3)=v_3(Y_1,Y_2,Y_3)$ yields, by the Theorem ($n=3$), that two 3-tuples (X_1,X_2,X_3) and (Y_1,Y_2,Y_3) are conjugate in F_m. Then the result follows from $X_1=X_3=X_4$ and $Y_1=Y_3=Y_4$.

Case II. $v_4(X_1,X_2,X_3,X_4)$ *is not a proper power.*

In view of Lemma 4, it follows that

$$\langle u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big),\ u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\rangle \text{ is non-cyclic.} \tag{4.9}$$

This enables us to apply Lemma 5 to the equality $v_4(X_1,\dots,X_4)=v_4(Y_1,\dots,Y_4)$: there exists a word $S\in F_m$ such that

$$\begin{aligned} 1&\neq u\big(v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\big)=Su\big(v_3(Y_1,Y_2,Y_3),v_3(Y_3,Y_2,Y_4)\big)S^{-1},\\ 1&\neq u\big(v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\big)=Su\big(v_3(Y_3,Y_2,Y_4),v_3(Y_4,Y_2,Y_1)\big)S^{-1}. \end{aligned} \tag{4.10}$$

Here, we have four subcases to consider.

Case II.1. *Both* $\langle v_3(X_1,X_2,X_3),v_3(X_3,X_2,X_4)\rangle$ *and* $\langle v_3(X_3,X_2,X_4),v_3(X_4,X_2,X_1)\rangle$ *are non-cyclic.*

The hypothesis of this case enables us to apply Lemma 5 to equalities (4.10): there exist words T_1 and T_2 in F_m such that

$$
\begin{aligned}
&1 \neq v_3(X_1, X_2, X_3) = T_1 v_3(Y_1, Y_2, Y_3) T_1^{-1}, \quad 1 \neq v_3(X_3, X_2, X_4) = T_1 v_3(Y_3, Y_2, Y_4) T_1^{-1}; \\
&1 \neq v_3(X_3, X_2, X_4) = T_2 v_3(Y_3, Y_2, Y_4) T_2^{-1}, \quad 1 \neq v_3(X_4, X_2, X_1) = T_2 v_3(Y_4, Y_2, Y_1) T_2^{-1}.
\end{aligned} \tag{4.11}
$$

Then by the Theorem ($n = 3$) applied to (4.11), there exist words U_1, U_2 and U_3 such that

$$
\begin{aligned}
&X_1 = U_1 Y_1 U_1^{-1}, \quad X_2 = U_1 Y_2 U_1^{-1}, \quad X_3 = U_1 Y_3 U_1^{-1}; \\
&X_3 = U_2 Y_3 U_2^{-1}, \quad X_2 = U_2 Y_2 U_2^{-1}, \quad X_4 = U_2 Y_4 U_2^{-1}; \\
&X_4 = U_3 Y_4 U_3^{-1}, \quad X_2 = U_3 Y_2 U_3^{-1}, \quad X_1 = U_3 Y_1 U_3^{-1}.
\end{aligned} \tag{4.12}
$$

Here, if one of $U_1 = U_2$, $U_2 = U_3$ and $U_3 = U_1$ is true, then the required result follows directly from (4.12). So assume that U_1, U_2 and U_3 are pairwise distinct. Combining the equalities in (4.12), we deduce that $\langle U_3^{-1}U_1, X_1\rangle$, $\langle U_1^{-1}U_2, X_2\rangle$, $\langle U_2^{-1}U_3, X_2\rangle$, $\langle U_3^{-1}U_1, X_2\rangle$, $\langle U_1^{-1}U_2, X_3\rangle$ and $\langle U_2^{-1}U_3, X_4\rangle$ are all cyclic, so that $\langle X_1, X_2\rangle$, $\langle X_3, X_2\rangle$ and $\langle X_4, X_2\rangle$ are cyclic. Since $\langle X_1, \dots, X_4\rangle$ is non-cyclic (this follows from $v_4(X_1, \dots, X_4) \neq 1$), we must have $X_2 = 1$; so, by (4.12), $Y_2 = 1$. Then by Lemma 10, the equalities on the first line of (4.11) yield that

$$
1 \neq v_2(X_3, X_1)^{400} = T_1 v_2(Y_3, Y_1)^{400} T_1^{-1}, \quad 1 \neq v_2(X_4, X_3)^{400} = T_1 v_2(Y_4, Y_3)^{400} T_1^{-1},
$$

namely,

$$
1 \neq v_2(X_3, X_1) = T_1 v_2(Y_3, Y_1) T_1^{-1}, \quad 1 \neq v_2(X_4, X_3) = T_1 v_2(Y_4, Y_3) T_1^{-1}. \tag{4.13}
$$

This is a similar situation to (3.10), so from here on, we can follow the proof of Case II.1 of the Theorem ($n = 3$) to obtain that two 3-tuples (X_1, X_3, X_4) and (Y_1, Y_3, Y_4) are conjugate in F_m. Since $X_2 = Y_2 = 1$, the desired result follows.

Case II.2. *$\langle v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\rangle$ is non-cyclic, and $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ is cyclic.*

In this case, we can apply Lemma 5 to the first equality of (4.10): there exists a word $V \in F_m$ such that

$$
1 \neq v_3(X_1, X_2, X_3) = V v_3(Y_1, Y_2, Y_3) V^{-1}, \quad 1 \neq v_3(X_3, X_2, X_4) = V v_3(Y_3, Y_2, Y_4) V^{-1}. \tag{4.14}
$$

Then the Theorem ($n = 3$) applied to (4.14) yields the existence of words W_1 and W_2 in F_m such that

$$\begin{aligned} &X_1 = W_1 Y_1 W_1^{-1}, \quad X_2 = W_1 Y_2 W_1^{-1}, \quad X_3 = W_1 Y_3 W_1^{-1}; \\ &X_3 = W_2 Y_3 W_2^{-1}, \quad X_2 = W_2 Y_2 W_2^{-1}, \quad X_4 = W_2 Y_4 W_2^{-1}. \end{aligned} \tag{4.15}$$

If $W_1 = W_2$, then the required result follows from (4.15). Now assume that $W_1 \neq W_2$. We deduce from (4.15) that $\langle W_1^{-1} W_2, X_2 \rangle$ and $\langle W_1^{-1} W_2, X_3 \rangle$ are cyclic, so that

$$\langle X_2, X_3 \rangle \text{ is cyclic.} \tag{4.16}$$

On the other hand, since $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1) \rangle$ is cyclic by the hypothesis of Case II.2, in view of Lemmas 10 and 11 ($n = 4$), we have that $v_3(X_3, X_2, X_4)$ is a proper power if and only if $v_3(X_4, X_2, X_1)$ is a proper power. For this reason, this case is reduced to the following three subcases.

Case II.2.1. $v_3(X_4, X_2, X_1) = 1$ *($v_3(X_3, X_2, X_4) \neq 1$ by the hypothesis of Case II.2).*

It follows from the Theorem ($n = 3$) that $\langle X_4, X_2, X_1 \rangle$ is cyclic. Since $\langle X_2, X_3 \rangle$ is cyclic by (4.16), in order to avoid a contradiction to the fact that $\langle X_1, \ldots, X_4 \rangle$ is non-cyclic, we must have

$$X_2 = 1; \tag{4.17}$$

thus $Y_2 = 1$ by (4.15). Then by Lemma 10, equalities (4.14) yield that

$$1 \neq v_2(X_3, X_1) = V v_2(Y_3, Y_1) V^{-1}, \quad 1 \neq v_2(X_4, X_3) = V v_2(Y_4, Y_3) V^{-1}.$$

This is the same situation as (4.13); hence from here on, repeating the proof of Case II.1, we obtain the desired result.

Case II.2.2. *Both $v_3(X_3, X_2, X_4)$ and $v_3(X_4, X_2, X_1)$ are proper powers.*

In view of Lemma 10, we have nine possibilities for the types of $v_3(X_3, X_2, X_4) \,\&\, v_3(X_4, X_2, X_1)$. Of these nine possible types, $(A1)\&(A1)$, $(A1)\&(A2)$, $(A1)\&(A3)$, $(A2)\&(A1)$, $(A2)\&(A3)$, $(A3)\&(A2)$

and $(A3)\&(A3)$ cannot occur, for if one of these occurred, then two of X_1, X_2, X_3 and X_4 should be equal to the empty word, which yields a contradiction to the non-triviality of $v_3(X_1, X_2, X_3)$, $v_3(X_3, X_2, X_4)$ or $v_3(X_4, X_2, X_1)$. So only $(A2)\&(A2)$ and $(A3)\&(A1)$ can actually occur.

If $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$ is of type $(A2)\&(A2)$, then $X_2 = 1$, which is the same situation as (4.17). Hence from here on, following the proof of Case II.2.1, we arrive at the desired result. If $v_3(X_3, X_2, X_4)\,\&\,v_3(X_4, X_2, X_1)$ is of type $(A3)\&(A1)$, then $X_4 = 1$. The result then follows from (4.15).

Case II.2.3. *Neither $v_3(X_3, X_2, X_4) \neq 1$ nor $v_3(X_4, X_2, X_1) \neq 1$ is a proper power.*

Since $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ is cyclic, it follows from Lemma 6 that $v_3(X_3, X_2, X_4) = v_3(X_4, X_2, X_1)$, so from the second equality of (4.10) that

$$1 \neq v_3(X_4, X_2, X_1)^{44} = Su\big(v_3(Y_3, Y_2, Y_4), v_3(Y_4, Y_2, Y_1)\big)S^{-1}. \tag{4.18}$$

This equality implies by Lemma 4 that $\langle v_3(Y_3, Y_2, Y_4), v_3(Y_4, Y_2, Y_1)\rangle$ is also cyclic. We then observe that equality (4.18) can hold only when neither $v_3(Y_3, Y_2, Y_4)$ nor $v_3(Y_4, Y_2, Y_1)$ is a proper power and $v_3(Y_3, Y_2, Y_4) = v_3(Y_4, Y_2, Y_1)$, by which (4.18) yields that

$$1 \neq v_3(X_4, X_2, X_1)^{44} = Sv_3(Y_4, Y_2, Y_1)^{44}S^{-1},$$

namely,

$$1 \neq v_3(X_4, X_2, X_1) = Sv_3(Y_4, Y_2, Y_1)S^{-1}.$$

Now apply the Theorem ($n = 3$) to this equality: there exists a word $W_3 \in F_m$ such that

$$X_4 = W_3Y_4W_3^{-1}, \quad X_2 = W_3Y_2W_3^{-1}, \quad X_1 = W_3Y_1W_3^{-1}.$$

Putting this together with (4.15), we have the same situation as (4.12) except that we already assumed $W_1 \neq W_2$ in Case II.2. Therefore, from here on, we can follow the proof of Case II.1 to derive the result.

Case II.3. *$\langle v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\rangle$ is cyclic, and $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ is non-cyclic.*

It is sufficient to repeat similar arguments to those in Case II.2 to arrive at the desired result.

Case II.4. *Both $\langle v_3(X_1, X_2, X_3), v_3(X_3, X_2, X_4)\rangle$ and $\langle v_3(X_3, X_2, X_4), v_3(X_4, X_2, X_1)\rangle$ are cyclic.*

Arguing as in the proof of Case II.2 of the Theorem ($n = 3$) replacing (3.8) and (3.9) by (4.9) and (4.10), respectively, we deduce that $v_3(X_3, X_2, X_4) = v_3(Y_3, Y_2, Y_4) = 1$. So

$$\langle X_3, X_2, X_4\rangle \text{ is cyclic;} \tag{4.19}$$

moreover, it follows from (4.10) that

$$1 \neq v_3(X_1, X_2, X_3)^{24} = Sv_3(Y_1, Y_2, Y_3)^{24}S^{-1}, \quad 1 \neq v_3(X_4, X_2, X_1)^{20} = Sv_3(Y_4, Y_2, Y_1)^{20}S^{-1},$$

namely,

$$1 \neq v_3(X_1, X_2, X_3) = Sv_3(Y_1, Y_2, Y_3)S^{-1}, \quad 1 \neq v_3(X_4, X_2, X_1) = Sv_3(Y_4, Y_2, Y_1)S^{-1}.$$

Then by the Theorem ($n = 3$) applied to these equalities, we have the existence of words Z_1 and Z_2 in F_m such that

$$\begin{aligned} X_1 &= Z_1Y_1Z_1^{-1}, \quad X_2 = Z_1Y_2Z_1^{-1}, \quad X_3 = Z_1Y_3Z_1^{-1}; \\ X_4 &= Z_2Y_4Z_2^{-1}, \quad X_2 = Z_2Y_2Z_2^{-1}, \quad X_1 = Z_2Y_1Z_2^{-1}. \end{aligned} \tag{4.20}$$

If $Z_1 = Z_2$, then the result follows from (4.20). Now assume that $Z_1 \neq Z_2$. Then equalities (4.20) yield that $\langle Z_1^{-1}Z_2, X_1\rangle$ and $\langle Z_1^{-1}Z_2, X_2\rangle$ are cyclic, so that $\langle X_1, X_2\rangle$ is cyclic. Since $\langle X_3, X_2, X_4\rangle$ is cyclic by (4.19), we must have $X_2 = 1$, which is the same situation as (4.17). Thus, from here on, we can follow the proof of Case II.2.1 to get the required result.

The Theorem ($n = 4$) is now completely proved. $\square$

5. The inductive step

In this section, we prove the inductive step of simultaneous induction which we use in Lemmas 11–13 and the Theorem. Let $n \geq 5$ throughout this section.

Lemma 11. *If both $v_{n-2}(X_1, \ldots, X_{n-2})$ and $v_{n-1}(Y_1, \ldots, Y_{n-1})$ are neither equal to the empty word nor proper powers, then $\langle v_{n-2}(X_1, \ldots, X_{n-2}), v_{n-1}(Y_1, \ldots, Y_{n-1})\rangle$ is non-cyclic.*

Proof. By way of contradiction, suppose that $\langle v_{n-2}(X_1, \ldots, X_{n-2}), v_{n-1}(Y_1, \ldots, Y_{n-1})\rangle$ is cyclic. Since both $v_{n-2}(X_1, \ldots, X_{n-2})$ and $v_{n-1}(Y_1, \ldots, Y_{n-1})$ are non-proper powers, it follows from Lemma 6 that

$$v_{n-2}(X_1, \ldots, X_{n-2}) = v_{n-1}(Y_1, \ldots, Y_{n-1}),$$

so from (1.3)–(1.4) and Lemmas 4 and 5 that there exists a word $S \in F_m$ such that

$$\begin{aligned} &u\big(v_{n-3}(X_1, \ldots, X_{n-3}), v_{n-3}(X_{n-3}, \ldots, X_{n-2})\big) = Su\big(v_{n-2}(Y_1, \ldots, Y_{n-2}), v_{n-2}(Y_{n-2}, \ldots, Y_{n-1})\big)S^{-1}, \\ &u\big(v_{n-3}(X_{n-3}, \ldots, X_{n-2}), v_{n-3}(X_{n-2}, \ldots, X_1)\big) = Su\big(v_{n-2}(Y_{n-2}, \ldots, Y_{n-1}), v_{n-2}(Y_{n-1}, \ldots, Y_1)\big)S^{-1}. \end{aligned} \tag{5.1}$$

We first assume that $\langle v_{n-3}(X_1, \ldots, X_{n-3}), v_{n-3}(X_{n-3}, \ldots, X_{n-2})\rangle$ is non-cyclic. This enables us to apply Lemma 5 to the first equality of (5.1): there exists a word $T \in F_m$ such that

$$\begin{aligned} v_{n-3}(X_1, \ldots, X_{n-3}) &= Tv_{n-2}(Y_1, \ldots, Y_{n-2})T^{-1} \\ v_{n-3}(X_{n-3}, \ldots, X_{n-2}) &= Tv_{n-2}(Y_{n-2}, \ldots, Y_{n-1})T^{-1}. \end{aligned} \tag{5.2}$$

If both sides of the first equality of (5.2) are non-proper powers, then this equality yields a contradiction to the induction hypothesis Lemma 11; if both sides of the first equality of (5.2) are proper powers, then we see from Lemma 10 and the induction hypothesis of Lemma 13 that they cannot be the same proper powers (because the exponents in both sides cannot be identical), a contradiction as well.

We next assume that $\langle v_{n-3}(X_1, \ldots, X_{n-3}), v_{n-3}(X_{n-3}, \ldots, X_{n-2})\rangle$ is cyclic. Then the first

equality of (5.1) yields by Lemma 4 that

$$\begin{aligned}(5.3)\quad &\langle v_{n-3}(X_1,\dots,X_{n-3}), v_{n-3}(X_{n-3},\dots,X_{n-2}),\\ &\qquad Sv_{n-2}(Y_1,\dots,Y_{n-2})S^{-1}, Sv_{n-2}(Y_{n-2},\dots,Y_{n-1})S^{-1}\rangle \text{ is cyclic.}\end{aligned}$$

Here, in view of the induction hypothesis of Lemma 13, we see that there are only two ways to avoid a contradiction to Lemma 8 and the induction hypothesis of Lemma 11: (i) any non-trivial word of $Sv_{n-2}(Y_1,\dots,Y_{n-2})S^{-1}$ and $Sv_{n-2}(Y_{n-2},\dots,Y_{n-1})S^{-1}$ is of type $(B4)$; (ii) any non-trivial word in (5.3) is of one of types $(B1)$, $(B2)$ and $(B3)$. (For $n=5$, there is only one way: any non-trivial word of $Sv_{n-2}(Y_1,\dots,Y_{n-2})S^{-1}$ and $Sv_{n-2}(Y_{n-2},\dots,Y_{n-1})S^{-1}$ is of one of types $(A1)$, $(A2)$ and $(A3)$.) However, in either case, we can observe that equalities (5.2) cannot hold. This contradiction completes the proof. $\square$

Lemma 12. *If* $u\big(v_{n-1}(X_1,\dots,X_{n-1}), v_{n-1}(Y_1,\dots,Y_{n-1})\big)=1$, *then* $v_{n-1}(X_1,\dots,X_{n-1})=1$ *and* $v_{n-1}(Y_1,\dots,Y_{n-1})=1$.

Proof. By Lemma 4, the hypothesis of the lemma yields that

$$v_{n-1}(X_1,\dots,X_{n-1})^6 = v_{n-1}(Y_1,\dots,Y_{n-1})^{-5}. \tag{5.4}$$

If one of $v_{n-1}(X_1,\dots,X_{n-1})$ and $v_{n-1}(Y_1,\dots,Y_{n-1})$ is equal to the empty word, then by (5.4) there is nothing to prove. So assume that $v_{n-1}(X_1,\dots,X_{n-1})\neq 1$ and $v_{n-1}(Y_1,\dots,Y_{n-1})\neq 1$. Since 6 and 5 are relatively prime, equality (5.4) implies that both $v_{n-1}(X_1,\dots,X_{n-1})$ and $v_{n-1}(Y_1,\dots,Y_{n-1})$ are either proper powers or non-proper powers. If both are proper powers, then a contradiction to equality (5.4) follows from the induction hypothesis of Lemma 13, for 6 times any of $960\cdot 400^{(n-4)}$, $400^{(n-3)}$, $576\cdot 400^{(n-4)}$ and 1936 cannot be equal to 5 times any of these. If both are non-proper powers, then, since $\langle v_{n-1}(X_1,\dots,X_{n-1}), v_{n-1}(Y_1,\dots,Y_{n-1})\rangle$ is cyclic, we have by Lemma 6 that $v_{n-1}(X_1,\dots,X_{n-1}) = v_{n-1}(Y_1,\dots,Y_{n-1})$. This together with (5.4) yields that $v_{n-1}(X_1,\dots,X_{n-1}) = v_{n-1}(Y_1,\dots,Y_{n-1}) = 1$. This contradiction to our assumption completes the proof. $\square$

Lemma 13. *Suppose that $\langle X_1, \dots, X_n \rangle$ is non-cyclic. Then $v_n(X_1, \dots, X_n) \neq 1$. Furthermore, either $v_n(X_1, \dots, X_n)$ is not a proper power or it has one of the following four forms:*

$$(B1)\ v_n(X_1,\dots,X_n) = \begin{cases} v_2(X_n, X_{n-1})^{960\cdot 400^{(n-3)}} & \text{and} \quad X_1 = X_2 = \cdots = X_{n-2} = 1, \quad \text{for } n \text{ even} \\ v_2(X_{n-1}, X_n)^{960\cdot 400^{(n-3)}} & \text{and} \quad X_1 = X_2 = \cdots = X_{n-2} = 1, \quad \text{for } n \text{ odd;} \end{cases}$$

$$(B2)\ v_n(X_1,\dots,X_n) = \begin{cases} v_2(X_1, X_n)^{400^{(n-2)}} & \text{and} \quad X_2 = X_3 = \cdots = X_{n-1} = 1, \quad \text{for } n \text{ even} \\ v_2(X_n, X_1)^{400^{(n-2)}} & \text{and} \quad X_2 = X_3 = \cdots = X_{n-1} = 1, \quad \text{for } n \text{ odd;} \end{cases}$$

$$(B3)\ v_n(X_1,\dots,X_n) = \begin{cases} v_2(X_{n-1}, X_1)^{576\cdot 400^{(n-3)}} & \text{and} \quad X_2 = \cdots = X_{n-2} = X_n = 1, \quad \text{for } n \text{ even} \\ v_2(X_1, X_{n-1})^{576\cdot 400^{(n-3)}} & \text{and} \quad X_2 = \cdots = X_{n-2} = X_n = 1, \quad \text{for } n \text{ odd;} \end{cases}$$

$$(B4)\ v_n(X_1,\dots,X_n) = v_{n-1}(X_1, X_2, \dots, X_{n-1})^{1936} \quad \text{and} \quad X_1 = X_{n-1} = X_n \neq 1.$$

Remark. *In view of Lemma 1 and the induction hypothesis of Lemma 13, $v_2(X_n, X_{n-1})$ and $v_2(X_{n-1}, X_n)$ in $(B1)$, $v_2(X_1, X_n)$ and $v_2(X_n, X_1)$ in $(B2)$, $v_2(X_{n-1}, X_1)$ and $v_2(X_1, X_{n-1})$ in $(B3)$, and $v_{n-1}(X_1, X_2, \dots, X_{n-1})$ in $(B4)$ are neither equal to the empty word nor proper powers.*

Proof. Closely follow the proof of Lemma 13 ($n = 4$) replacing references to Lemmas 10, 11 ($n = 4$), 12 ($n = 4$) and the Theorem ($n = 3$) by references to the induction hypothesis of Lemma 13, Lemmas 8 and 11, Lemma 12 and the induction hypothesis of the Theorem, respectively. Different situations from Lemma 13 ($n = 4$) can possibly occur only in Case III.2 and Case III.5, which we reconsider below.

Case III.2. *$\langle 1 \neq v_{n-1}(X_{n-1}, X_2, X_3, \dots, X_{n-2}, X_n),\ 1 \neq v_{n-1}(X_n, X_2, X_3, \dots, X_{n-2}, X_1) \rangle$ is cyclic, and $v_{n-1}(X_1, X_2, \dots, X_{n-1}) = 1$.*

Since $v_{n-1}(X_1, X_2, \dots, X_{n-1}) = 1$, we have, by the induction hypothesis of the Theorem, that

$$\langle X_1, X_2, \dots, X_{n-1} \rangle \ \text{ is cyclic.} \tag{5.5}$$

Also since $\langle 1 \neq v_{n-1}(X_{n-1}, X_2, \dots, X_{n-2}, X_n), 1 \neq v_{n-1}(X_n, X_2, \dots, X_{n-2}, X_1) \rangle$ is cyclic, in view of Lemmas 8 and 11 and the induction hypothesis of Lemma 13, this case is reduced to the following two cases:

(i) both $v_{n-1}(X_{n-1}, X_2, \dots, X_{n-2}, X_n)$ and $v_{n-1}(X_n, X_2, \dots, X_{n-2}, X_1)$ are proper powers;

(ii) neither $v_{n-1}(X_{n-1}, X_2, \dots, X_{n-2}, X_n)$ nor $v_{n-1}(X_n, X_2, \dots, X_{n-2}, X_1)$ is a proper power.

Case (i) is divided further into subcases according to types of $v_{n-1}(X_{n-1}, X_2, \ldots, X_{n-2}, X_n)$ & $v_{n-1}(X_n, X_2, \ldots, X_{n-2}, X_1)$ by the induction hypothesis of Lemma 13. However, the former word cannot be of type $(B3)$ nor $(B4)$, for this type together with (5.5) yields a contradiction to the hypothesis of Lemma 13. For the same reason, the latter cannot be of type $(B1)$ nor $(B4)$. Also $(B1)$&$(B2)$ and $(B2)$&$(B3)$ cannot occur, for these types yield a contradiction to the non-triviality of $v_{n-1}(X_{n-1}, X_2, \ldots, X_{n-2}, X_n)$ and $v_{n-1}(X_n, X_2, \ldots, X_{n-2}, X_1)$, respectively. Thus, in Case (i), only $(B1)$&$(B3)$ and $(B2)$&$(B2)$ need to be considered. This allows us to follow the proof of Lemma 13 ($n = 4$) from here on.

Case III.5. $\langle\ 1 \neq v_{n-1}(X_1, X_2, \ldots, X_{n-1}),\ 1 \neq v_{n-1}(X_{n-1}, X_2, X_3, \ldots, X_{n-2}, X_n),\ 1 \neq v_{n-1}(X_n, X_2, X_3, \ldots, X_{n-2}, X_1)\rangle$ *is cyclic.*

In this case, in order to be able to keep following the proof of Lemma 13 ($n = 4$), it is sufficient prove the following:

Claim. $1 \neq v_{n-1}(X_1, X_2, \ldots, X_{n-1}) = v_{n-1}(X_{n-1}, \ldots, X_{n-2}, X_n) = v_{n-1}(X_n, \ldots, X_{n-2}, X_1)$.

Proof of the Claim. If none of these is a proper power, then the assertion follows immediately from Lemma 6. So assume that one of these is a proper power. Then, in view of the induction hypothesis of Lemma 13 and Lemmas 8 and 11, the other two also have to be proper powers. Here, in order to avoid a contradiction to the non-triviality of these words, all of these words must have the same type either $(B2)$ or $(B4)$. The treatment of $(B2)$ is the same as in the Theorem ($n = 4$); if all of these words are of type $(B4)$, then we have $X_1 = X_{n-2} = X_{n-1} = X_n \neq 1$, which proves the claim. $\square$

The proof of Lemma 13 is complete. $\square$

Proof of the Theorem. The additional property that $v_n(X_1, \ldots, X_n) = 1$ if and only if the subgroup $\langle X_1, \ldots, X_n\rangle$ of F_m is cyclic follows immediately from definition (1.4) of $v_n(x_1, \ldots, x_n)$ and Lemma 13. Now supposing $1 \neq v_n(X_1, \ldots, X_n) = v_n(Y_1, \ldots, Y_n)$, we want to prove that there

exists a word $Z \in F_m$ such that

$$Y_i = ZX_iZ^{-1} \quad \text{for all} \quad i = 1, \ldots, n,$$

that is, we want to show that $v_n(x_1, \ldots, x_n)$ is a C-test word. From here on, repeat the proof of the Theorem ($n = 4$) replacing references to Lemmas 10, 11 ($n = 4$), 13 ($n = 4$) and the Theorem ($n = 3$) by references to Lemma 13, Lemmas 8 and 11, Lemma 13 and the induction hypothesis of the Theorem, respectively. A different situation from the Theorem ($n = 4$) can possibly occur only in Case II.2.2, which we reconsider below.

Case II.2.2. *Both* $v_{n-1}(X_{n-1}, X_2, X_3, \ldots, X_{n-2}, X_n)$ *and* $v_{n-1}(X_n, X_2, X_3, \ldots, X_{n-2}, X_1)$ *are proper powers.*

In view of Lemma 13, $v_{n-1}(X_{n-1}, X_2, \ldots, X_{n-2}, X_n)$ & $v_{n-1}(X_n, X_2, \ldots, X_{n-2}, X_1)$ has 16 possible types, of which it suffices to consider the types involving $(B4)$, namely, $(B1)$&$(B4)$, $(B2)$&$(B4)$, $(B3)$&$(B4)$, $(B4)$&$(B4)$, $(B4)$&$(B1)$, $(B4)$&$(B2)$ and $(B4)$&$(B3)$, since the consideration for the remaining types is the same as in the proof of the Theorem ($n = 4$). However, none of these types except for $(B4)$&$(B4)$ can actually occur, for these types except for $(B4)$&$(B4)$ together with the hypothesis of Case II.2 that $\langle v_{n-1}(X_{n-1}, X_2, \ldots, X_{n-2}, X_n), v_{n-1}(X_n, X_2, \ldots, X_{n-2}, X_1)\rangle$ is cyclic yield a contradiction to Lemma 8.

If $(B4)$&$(B4)$ occurs, then the equality corresponding to the second one of (4.10), namely,

$$\begin{aligned} 1 \neq u\big(&v_{n-1}(X_{n-1}, X_2, \ldots, X_{n-2}, X_n), v_{n-1}(X_n, X_2, \ldots, X_{n-2}, X_1)\big) \\ &= Su\big(v_{n-1}(Y_{n-1}, Y_2, \ldots, Y_{n-2}, Y_n), v_{n-1}(Y_n, Y_2, \ldots, Y_{n-2}, Y_1)\big)S^{-1} \end{aligned}$$

forces $v_{n-1}(Y_{n-1}, Y_2, \ldots, Y_{n-2}, Y_n)\,\&\,v_{n-1}(Y_n, Y_2, \ldots, Y_{n-2}, Y_1)$ to have type $(B4)$&$(B4)$ as well. Thus, we have

$$X_1 = X_{n-2} = X_{n-1} = X_n \neq 1 \quad \text{and} \quad Y_1 = Y_{n-2} = Y_{n-1} = Y_n \neq 1,$$

which together with

$$X_{n-1} = W_2 Y_{n-1} W_2^{-1}, \quad X_2 = W_2 Y_2 W_2^{-1}, \quad \cdots, \quad X_{n-2} = W_2 Y_{n-2} W_2^{-1}, \quad X_n = W_2 Y_n W_2^{-1}$$

corresponding to the equalities on the second line of (4.15) implies the desired result. This completes Case II.2.2. □

6. Proof of Corollary 2

We are now in a position to prove Corollary 2.

Proof of Corollary 2. Let ϕ and ψ be endomorphisms of F_m with non-cyclic images. Take

$$u_1 = v_m(x_1, x_2, \ldots, x_m) \quad \text{and} \quad u_2 = v_{m+1}(x_{m-1}, x_{m-2}, \ldots, x_1, x_m, x_1).$$

Supposing $\phi(u_i) = \psi(u_i)$, for $i = 1, 2$, we want to prove $\phi = \psi$. By Corollary 1, the equality $\phi(u_1) = \psi(u_1)$ implies that

$$\phi = \tau_S \circ \psi, \quad \text{where} \ \ \langle S, \psi(u_1)\rangle \ \ \text{is cyclic.} \tag{6.1}$$

Here, it is sufficient to show $S = 1$ to draw the desired result. By (6.1), the other equality $\phi(u_2) = \psi(u_2)$ yields that

$$\psi(u_2) = \phi(u_2) = S\psi(u_2)S^{-1},$$

so that $\langle S, \psi(u_2)\rangle$ is cyclic. Then $S = 1$ follows obviously from the following:

Claim. *$\langle \psi(u_1), \psi(u_2)\rangle$ is non-cyclic.*

Proof of the Claim. Putting $X_i = \psi(x_i)$, for all $i = 1, 2, \ldots, m$, the claim is equivalent to

$$\langle v_m(X_1, X_2, \ldots, X_m), v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1)\rangle \ \ \text{is} \ \ \text{non-cyclic.}$$

Since ψ has non-cyclic image, $\langle X_1, X_2, \ldots, X_m\rangle$ is non-cyclic. This implies by the Theorem that

$$v_m(X_1, X_2, \ldots, X_m) \neq 1 \ \ \text{and} \ \ v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1) \neq 1.$$

We treat two cases separately.

Case I. $v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1)$ *is not a proper power.*

In this case, $v_m(X_1, X_2, \ldots, X_m)$ cannot be of form $(B4)$, for if $v_m(X_1, X_2, \ldots, X_m)$ were of form $(B4)$ then $X_1 = X_{m-1} = X_m \neq 1$, which forces $v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1)$ to be a proper power of form $(B4)$, a contradiction. Then the claim follows from Lemmas 8 and 11.

Case II. $v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1)$ *is a proper power.*

This case can occur only when $m \geq 3$, for if $v_3(X_1, X_2, X_1)$ were a proper power, then we should have $X_2 = 1$, hence $\langle X_1, X_2 \rangle$ is cyclic, contrary to our assumption that ψ has non-cyclic image. It then follows from Lemma 13 that $v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1)$ has one of four types $(B1)$–$(B4)$. Of these types, $(B1)$ cannot occur, for $(B1)$ implies $X_{m-1} = X_{m-2} = \cdots = X_1 = 1$, contrary to the fact that $\langle X_1, X_2, \ldots, X_m \rangle$ is non-cyclic. For a similar reason, $(B2)$ cannot occur, either. On the other hand, $(B4)$ cannot occur when $m = 3$, for $(B4)$ together with $m = 3$ implies $X_2 = X_3 = X_1 \neq 1$, contrary to $\langle X_1, X_2, X_3 \rangle$ being non-cyclic. When $m \geq 4$, $(B4)$ implies that $X_{m-1} = X_m = X_1 \neq 1$, so that $v_m(X_1, X_2, \ldots, X_m)$ has type $(B4)$ as well. Then the claim follows from Lemma 11.

It remains to consider type $(B3)$. If $v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1)$ has type $(B3)$, then we have that

$$v_{m+1}(X_{m-1}, X_{m-2}, \ldots, X_1, X_m, X_1) = \begin{cases} v_2(X_{m-1}, X_m)^{576 \cdot 400^{(m-2)}}, & \text{for } m \text{ even} \\ v_2(X_m, X_{m-1})^{576 \cdot 400^{(m-2)}}, & \text{for } m \text{ odd,} \end{cases}$$

$$\text{and} \quad X_{m-2} = X_{m-3} = \cdots = X_1 = 1. \tag{6.2}$$

Then (6.2) forces $v_m(X_1, X_2, \ldots, X_m)$ to have type $(B1)$; hence

$$v_m(X_1, X_2, \ldots, X_m) = \begin{cases} v_2(X_m, X_{m-1})^{960 \cdot 400^{(m-3)}}, & \text{for } m \text{ even} \\ v_2(X_{m-1}, X_m)^{960 \cdot 400^{(m-3)}}, & \text{for } m \text{ odd.} \end{cases}$$

Now, by way of contradiction, suppose the contrary of the claim. It then follows that

$$\langle v_2(X_{m-1}, X_m), v_2(X_m, X_{m-1}) \rangle \text{ is cyclic,}$$

so by Lemmas 1 and 3 that

$$1 \neq v_2(X_{m-1}, X_m) = v_2(X_m, X_{m-1}).$$

Applying Lemma 2 to this equality yields the existence of $T \in F_m$ such that

$$X_{m-1} = TX_mT^{-1} \quad \text{and} \quad X_m = TX_{m-1}T^{-1}. \tag{6.3}$$

Combining these equalities, we see that both $\langle X_{m-1}, T\rangle$ and $\langle X_m, T\rangle$ are cyclic. Here, if $T \neq 1$, then $\langle X_{m-1}, X_m\rangle$ is cyclic; if $T = 1$, then $X_{m-1} = X_m$ by (6.3). This together with (6.2) yields a contradiction to the fact that $\langle X_1, X_2, \ldots, X_m\rangle$ is non-cyclic, which completes the proof of the claim. □

The proof of Corollary 2 is completed. □

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